Instructor: Girish Varma • Course Code: IMA410 Solve 2 problems • 5 marks each

1 Averaging Argument

1. Suppose there are N members in a library containing B books. If every person likes at least $1/3$ of the books in the library, show that the library has a book, which at least $1/3$ of people like. (3 marks)

2. Let $f : \{0,1\}^n \to \{0,1\}$ be some function. If we have an algorithm A which take inputs $x \in \{0,1\}^n, y \in \{0,1\}^m$ such that $A(x,y) = f(x)$ with probability at least ρ where x is chosen at random and y is chosen randomly, independently from $\{0,1\}^m$. Show that there exists a single string $y^* \in \{0,1\}^m$ such that $\Pr_x[A(x,y^*) = f(x)] \geq \rho$ (ie. only x is chosen randomly here). (2 marks)

2 Hypergraphs and Rainbow Colorings

A *k-uniform hypergraph* is a generalization of a graph $\mathcal{G}([n],\mathcal{E})$, which has a vertex set $[n]$ and a *hyperedge* set $\mathcal{E} = \{e_1, \cdots, e_m\}$ where each $e_i \subseteq [n]$ of size k. Note that for $k = 2$, we get an undirected graph. An assignment of colors to the vertex set is said to (t,c)*-rainbow color* an edge e, if it uses only c distinct colors in total and the number of distinct colors in e is at least t. A (t,c)*-rainbow coloring* of hypergraph $\mathcal{G}([n],\mathcal{E})$ is an assignment that is a (t,c)-rainbow coloring of every $e \in E$.

1. Show that for a random assignment of colors in $[k]$ to the vertices, the expected number of (k,k)-rainbow colored edges is $\frac{|\mathcal{E}|\cdot k!}{k^k}$ (2 marks)

2. Show that any hypergraph with number of edges $m < \frac{k^k}{k^k - k!}$, has a (k,k)-rainbow coloring. (1 mark)

3. Use the method of conditional expectation, to give a deterministic algorithm to find the (k,k)-rainbow coloring for $\frac{|\mathcal{E}|\cdot k!}{k^k}$ from first part. (2 marks)

(You can substitute $k = 2$ and solve all the problems to get 2.5 marks).

3 Randomized Perfect Matching

Consider a bipartite graph $G(V, E)$ with vertex set $V = [n]$. Let A be the adjacency matrix. Let

$$\text{perm}(A) = \sum_{\sigma \in S_n} \left(\prod_{i \in [n]} a_{i\sigma(i)} \right) \quad \text{and} \quad \det(A) = \sum_{\sigma \in S_n} (-1)^{\text{parity}(\sigma)} \left(\prod_{i \in [n]} a_{i\sigma(i)} \right)$$

1. Show that $\text{perm}(A)$ is equal to the number of perfect matchings in G. (1 mark)
2. Let B be the matrix obtained from A by replacing A_{ij} with $x_{ij}A_{ij}$ where x_{ij} are some variables. Note that $\det(B), \text{perm}(B)$ are polynomials in these variables. Show that $\det(B) \equiv 0$ if and only if there are no perfect matching. (1 marks)
3. Let B' be the random matrix obtained by substituting each x_{ij} with uniformly and independently chosen values from $[2n]$.
   Show that $\Pr[\det(B') = 0 | B \not\equiv 0] \leq 1/2$. (2 marks)
4. Using the above give a randomized algorithm with one-sided error for checking whether a graph has perfect matching. (1 mark)
