Def: (Deciding)

$L \subseteq \{0,1\}^*$ (a language)

is **decided** by a TM (3 tape TM) (3 heen)

$M = (Q, \Gamma, \delta)$ if

$\forall x \in L, \quad M(x) = 1$

$\forall x \notin L, \quad M(x) = 0$

- Decision / Search
- Lang / Bool fun
- Turing Machine
- Binary Encoding
  - Tuples
  (Prefix free encoding)
- TM can be encoded

M solves decision problem

All languages have a TM that decides it?

- Any TM M can be encoded to $\{0,1\}^*$

$E : \mathcal{M} \longrightarrow \{0,1\}^*$

# of TM's are countably infinite.

$\{0,1\}^* \longleftrightarrow \mathbb{N}$ bijection

How many languages $L \subseteq \{0,1\}^*$?

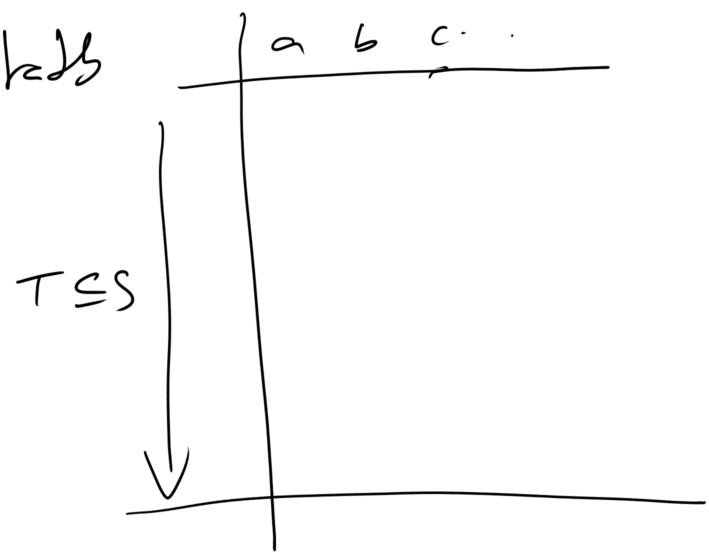
Cantor: Diagonalization

For any set S,

there is no bijection between

S and $\mathcal{P}(S)$

of L are uncountably infinite.



→ NO!

Halting Problem

Input: $\langle M, x \rangle \in \{0,1\}^*$

Output: 1 if M halts on x

0 if M does not halt on x

$L_{HALT} = \{ y : \text{s.t. } y = \langle M, x \rangle$ and M halts on $x \}$

Suppose M^* decides L_{HALT}

Universal Turing Machine.

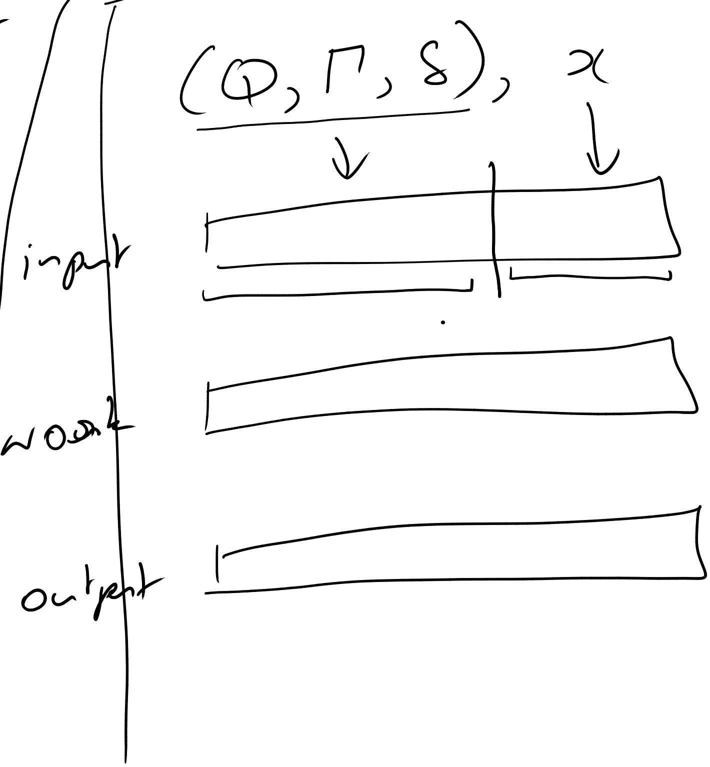
Let D be a TM

input: $\langle M \rangle \in \{0,1\}^*$

Run M on $\langle M \rangle$

- If M halts on $\langle M \rangle$, then infinite loop
- If M does not halt on $\langle M \rangle$ then output 1

can be done using M^*



Universal Turing Machine] Interpreter.

- Input: <M, x>
- Output: M(x)

$\{0,1\}^*$

Does D halt on <D>?

- If D halts <D>,
  the D will loop on <D>
- If D does not halt on <D>,
  then D halts and outputs 1

⇓

- D does not exist.
- If M^* exist then D also exist.
- M^* does not exist.
- There is not TM that decides L_{HALT}.

Recursive Langs

= { L : There is a TM that decides L }

HAM-CYCLE $\in$ Recursive

L_{HALT} $\notin$ Recursive.

Recursively Enumerable (RE)

$L \in RE$ if there is a TM M s.t.

- $\forall x \in L$, $M(x) = 1$
- $\forall x \notin L$, $M(x) = 0$ or M loops.

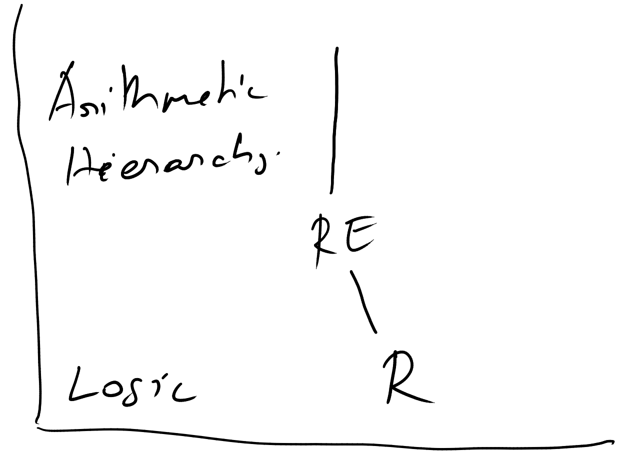
$L_{HALT} \in RE$

$M_{HALT}(\langle M, x \rangle)$

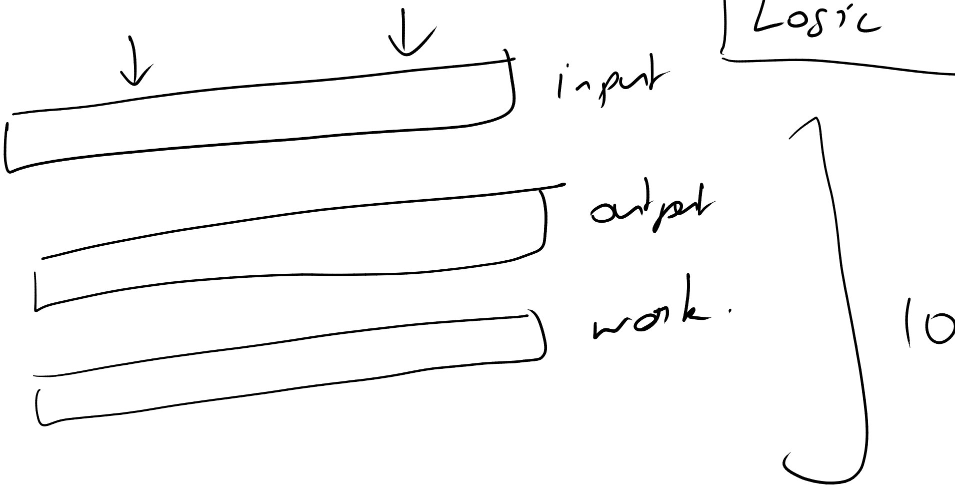
Run M on x
If it halts
output 1

Are there languages $\notin RE$?

Robustness of TMs.

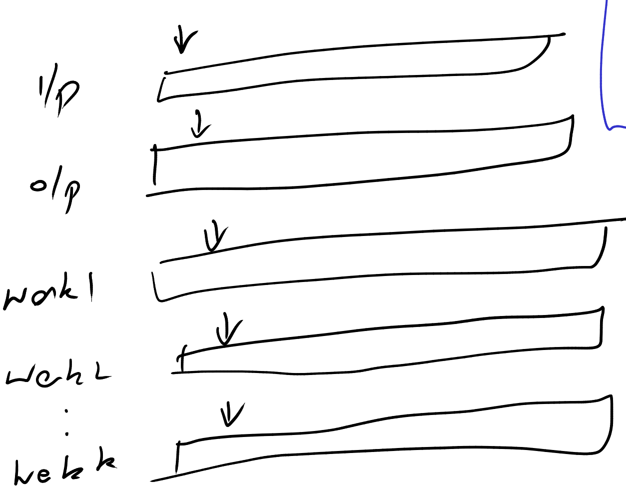


Q, Γ, δ.



Suppose the TM had more tapes or more heads or more alphabet ?

will definition of decidability change ?



$$\delta : Q \times \Gamma^{k+1} \longrightarrow Q \times \Gamma^{k+1} \times \{L, R, S\}^{k+2}$$

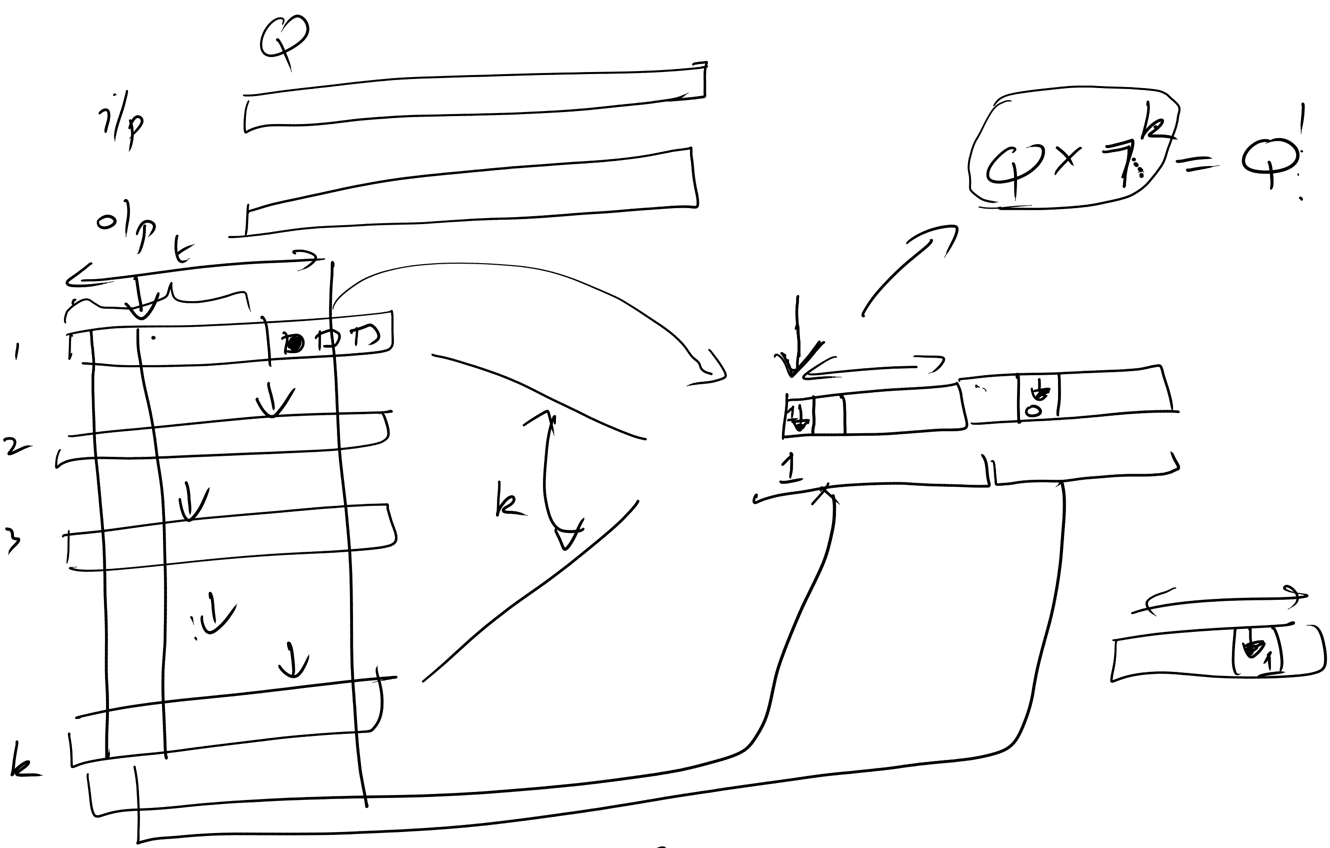
Γ^{k+1} ↑ reading. Γ^{k+1} ↑ writing. $\{L, R, S\}^{k+2}$ ↑ moving.

$Q,$

Suppose L has a k-tape TM

deciding it then

is there a 3 tape TM for it?

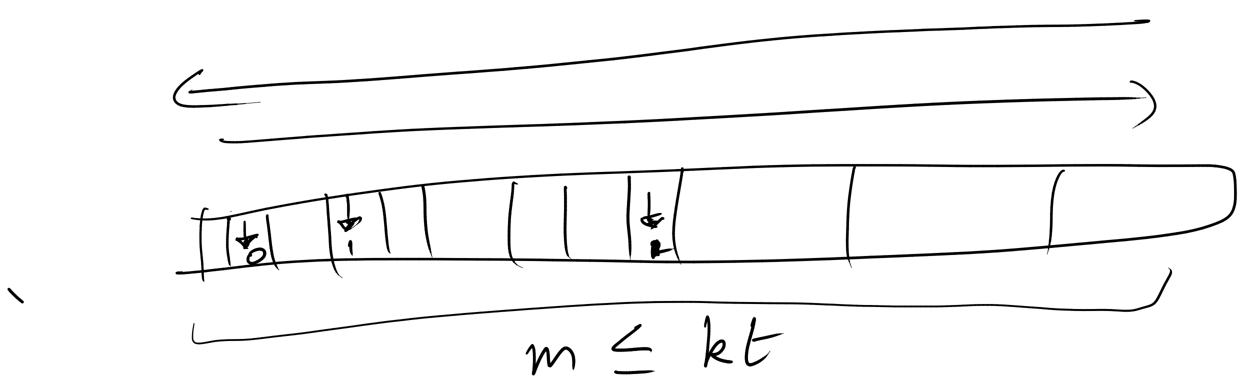


$\Gamma' = \Gamma \times \{\downarrow, \cdot\}$ Finite?

$\{0, 1, \square, \triangleright\}$

$\{0\cdot, 0\downarrow, 1\cdot, 1\downarrow, \square\cdot, \square\downarrow, \triangleright\cdot, \triangleright\downarrow\}$

To simulate 1 step of k tape TM, we do 2 passes of 1-Tape TM over entire work tape.



Suppose running time of k-tapes is $\underline{t}$.

What is running time of 1-tape?

k tape	1 tape
1 step	$2m \le 2kt$
t steps	$t \times 2kt$
	$2kt^2$ steps.
↑	↑