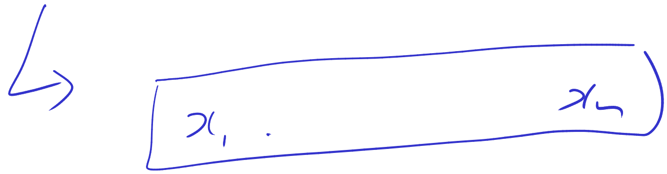
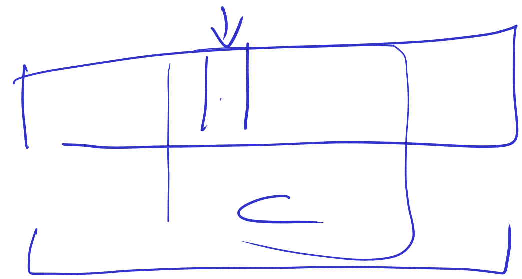
Previously:

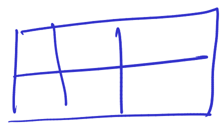
- P, NP
  ↳ verifier TM (alternate def)
- Reductions, NP complete.
- Cook-Levin Theorem
  3SAT / 3CNF is NP-complete.
- More Reductions:
  HAM-PATH, INDSET, VERTEX COVER, INTEGER-PROG, $\in$ NP-complete.

$$\varphi_{stat} \wedge \underbrace{\varphi_{cell} \wedge \varphi_{move} \wedge \varphi_{acc}}$$





x



$$x \longrightarrow \varphi$$

$$y_{1 i x_i} = 1 \qquad \underline{\{ y_{1\, n+t\, a} \}}_{a \in Q \cup \Gamma}$$

- Tautology

$= \{ \langle \varphi \rangle : \varphi$ is a CNF formula s.t. $\forall a, \varphi(a) = 1 \}$

$$(\underline{x_1 \vee \neg \check{x_1}} \vee x_5) \wedge (\underline{x_3 \vee \neg \check{x_3}} \vee x_7)$$

$$\wedge (\quad) \wedge (\quad)$$

- Is Tautology $\in$ P, NP, DTIME, DSPACE?

$\overline{\text{Tautology}} = \{ \langle \varphi \rangle : \varphi$ is a CNF s.t. $\exists a\ \varphi(a) = 0 \}$

$\parallel$

$\{0,1\}^* \setminus$ Tautology.

$\varphi \in \overline{\text{Tautology}} \iff \neg\varphi \in \text{CNF}$

$$\text{Tautology} \equiv \overline{\text{CNF}} = \{0,1\}^* \setminus \text{CNF}$$

$$\text{No-HAMPATH} = \{\langle G \rangle : G \text{ does not have a ham. path}\}$$

$$\text{HAMPATH} = \{0,1\}^* \setminus \text{No-HAMPATH}$$

NP

Def: (CoNP)

$L \in \text{CoNP}$ if $\exists$ a poly time TM $V(\cdot,\cdot)$ s.t.

$\forall x \in L, \quad \forall c, \ V(x,c) = 1$

$\forall x \notin L, \quad \exists c, \ V(x,c) = 0$

- Tautology $\in$ CoNP

  V (φ, c) {

  Evaluate φ on c

  and return output.

  }

- No-HAMPATH $\in$ CoNP

  a path in the graph

  V (<G>, p) {

  - Check if p is a path in G
  - Check if p is a HAMPATH
  - If checks fail return 1
  - Otherwise return 0

  }

- It is open even to find a NTM (poly time) for Tautology, No HAMPATH.

- <u>Def (CoNP-Complete)</u>

  $L \in$ CoNP-Complete if $\forall L' \in \text{CoNP}, L' \leq L$.

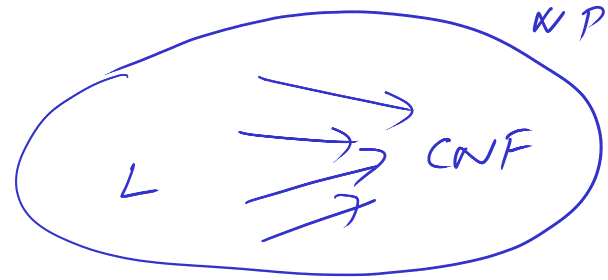
- If $L \in$ CoNP-Complete $\cap$ P

  $\Rightarrow$ CoNP $=$ P

- If $L \in$ CoNP-Complete $\cap$ NP

  $\Rightarrow$ CoNP $=$ NP

$\overline{L} = \{0,1\}^* \setminus L$



$\overline{L} \longrightarrow \overline{CNF}$

By Cook-Levin,

$\forall L \in NP,$

$$L \le CNF$$

$$\overline{L} \le \overline{CNF}$$

Cook-Levin Reduction on $x \notin L$,

$$x \longmapsto \varphi \notin CNF$$

$\boxed{x \in L \iff \varphi \in CNF}$

$\Updownarrow$

$\Rightarrow \varphi \in \overline{CNF}$

$\boxed{x \in \overline{L} \iff \varphi \in \overline{CNF}}$

$\Rightarrow \overline{CNF} \in$ CoNP-Complete

$x \in L \Rightarrow \varphi \in CNF$ and

$x \notin L \Rightarrow \varphi \notin CNF$

$x \notin \overline{L} \Rightarrow \varphi \notin \overline{CNF}$

$x \in \overline{L} \Rightarrow \varphi \in \overline{CNF}$

$$\boxed{\overline{CNF} = \{0,1\}^* \setminus CNF}$$

$\overline{HAM\text{-}PATH} \in$ CoNP-Complete.

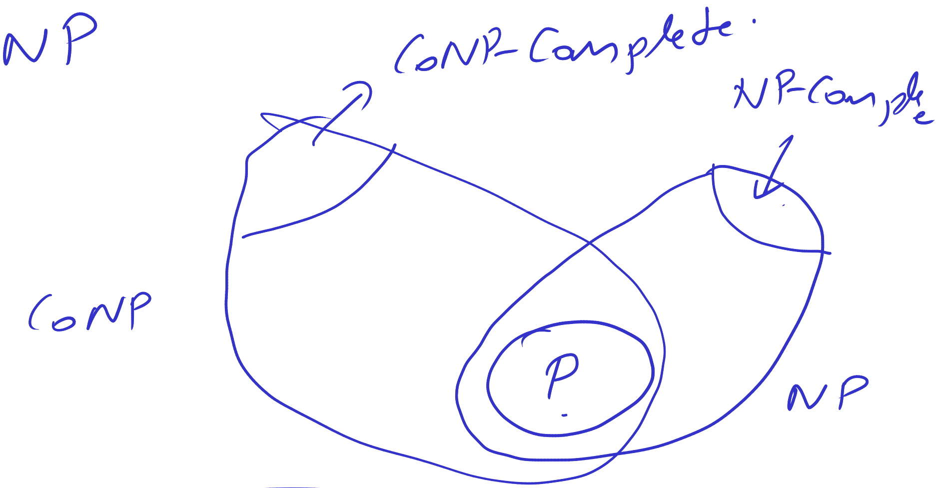
$L \in$ NPComplete $\Leftrightarrow \overline{L} \in$ CoNPComplete.

Open Problem $NP = CoNP$?

Can there be $L \in NP \cap CoNP$?

$P \subseteq CoNP$?

- $P \subseteq CoNP \cap NP$



$DTIME(n^k) \subseteq DSPACE(n^k)$

Claim

$DSPACE(n^k) \subseteq DTIME(2^{n^k})$

$\downarrow$ M

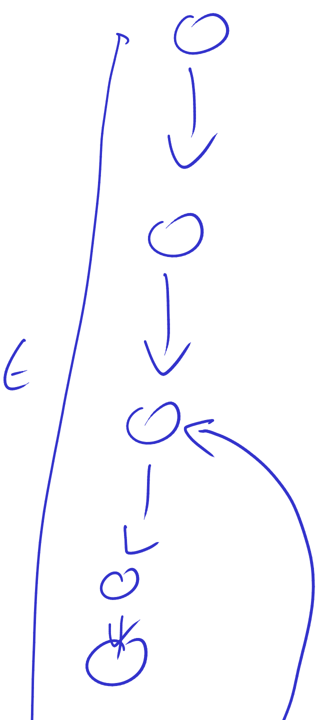
Proof:

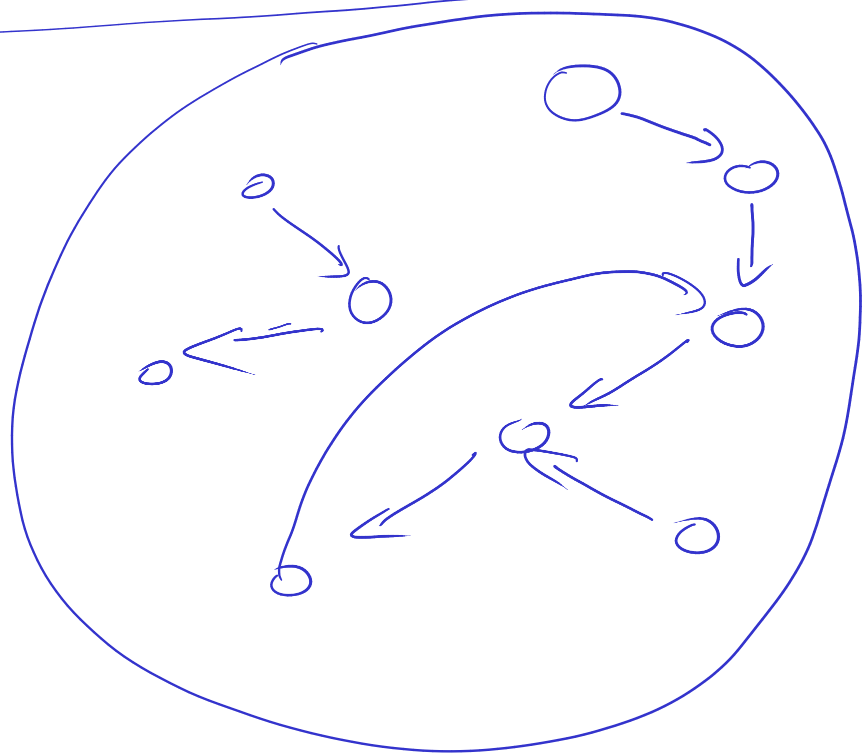
bit to write config of M, $O(n^k)$

of configurations $\leq 2^{O(n^k)} = t$

Claim: Running time of $M \leq t$

Proof: Suppose running time of $M > t$.





$\Rightarrow M$ doesn't halt.

$\rightarrow \leftarrow$

$\boxtimes$

REACHABILITY

$= \{ \langle G, s, t \rangle : t$ is reachable from s in $G \}$

$\in P$

$\in DSPACE(n \log(n))$

$\downarrow$

<u>Claim:</u>

REACHABILITY $\in$ DSPACE$(\log^2 n)$

<u>Proof:</u>

```
reach(s, t, l) {
 // checks if s-t path of length l
  for v ∈ V:
     check reach(s, v, l/2)
     check reach(v, t, l/2)
     if both checks pass return 1
} return 0
```

l max value $= n$.

depth of stack $= \log(n)$

$$\boxed{S(n) = \cancel{2}\, S(n/2) + \log n \quad ?}$$

$$S(n) = S(n/2) + \log n$$

$$S(n) = O(\log^2 n)$$

— We can reuse the space

$$T(n) \le 2^{O(\log^2 n)} = 2^{(\log n)(\log n)}$$

$$= \left(2^{\log n}\right)^{\log n} = n^{\log n}$$

(not poly time)

Open Problem

Design algo for Reachability.

- with $O(\log n)$ space.
  (know for undirected graphs Reingold (2002)
  (Godel Prize))

- with $O((\log n)^k)$ space
  with running time
  polynomial ie (n^{50})