Previously

$M \in \mathbb{R}^{n \times n}$

Properties of Symmetric Real Matrices

1. Eigenvalues are real.
2. Eigenvectors corresponding to different eigenvalues are orthogonal.
3. If M has an O.N.E.V.B (orthonormal, eigenvectors, basis) then M is symmetric.
4. If P is the change of basis matrix from one ONB to another ONB then $P^{-1} = P^T$.
5. If M has n distinct eigenvalues $\Rightarrow$ M has an ONEV.B

Spectral Theorem

M is symmetric $\iff$ M has ONEVB

Theorem:

M is symmetric $\implies$ M has ONEVB

Proof:

The char polynomial of M has atleast 1 root. If M is symmetric the it has to be real.

λ_1, w_1

$\uparrow$ $\mathbb{R}$ $\quad$ $\uparrow$ $\mathbb{R}^n$

$\in \mathbb{R}$

$M w_1 = \lambda w_1$ (whichever is nonzero)

$w_1' = Re(w_1)$ or $Im(w_1)$

$M(Re(w_1) + i\, Im(w_1))$

$= \lambda(\quad \quad)$

$\therefore$ w_1' also has eigenvalue λ_1

We will find O.N.B for $\mathbb{R}^n$

$$\{ w_1', \underbrace{\underbrace{w_2' \cdots\ w_n'}}_{i} \}$$

basis for $w_1'^{\perp}$

$\dim(w_1'^{\perp}) = \underline{n-1}$

P - change of basis matrix

$$M' = P^{-1} M P = P^T M P$$

$$M' \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$P^{-1} = P^T$

$\Downarrow$

$$M' = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & & & & \\ 0 & & \boxed{\tilde{M}'} & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix}$$

$$(M')^T = P^T M P = M'$$

$$\begin{bmatrix} 0 \\ a \\ \vdots \\ \vdots \end{bmatrix}$$

$\tilde{M}' \in \mathbb{R}^{n-1 \times n-1}$

$\tilde{M}'$ is symmetric

$\downarrow$

(by mathematical induction)

$$M'' = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & & \\ & 0 & & \boxed{\tilde{M}''} & & \\ 0 & 0 & & & & \end{bmatrix}$$

$\tilde{M}'' \in \mathbb{R}^{n-2 \times n-2}$

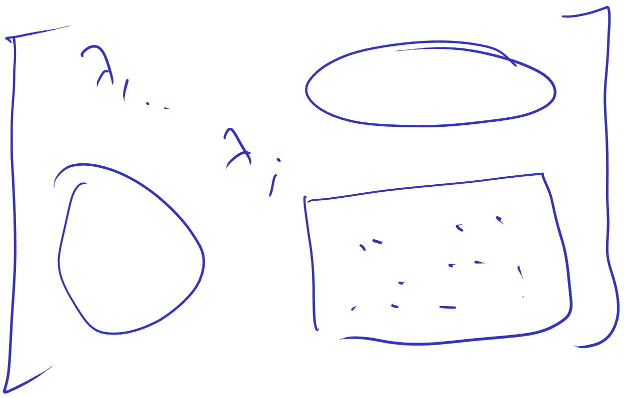
symmetric

$\longrightarrow \quad \longrightarrow \quad \longrightarrow \quad \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ 0 & 0 & & \lambda_n \end{bmatrix} = P^{-1}MP$

$w_1^{(2)} = w_1^{(1)}$

in some O.N.B $\quad w_1^{(1)} \cdots w_n^{(1)}$

$\longrightarrow$ O.N.E.V.B $\quad w_1^{(n)} \cdots w_n^{(n)}$



$\{ \underbrace{w_1^{(i)}, w_2^{(i)}, \ldots w_i^{(i)}}_{\text{e.vectors.}}, \underbrace{w_{i+1}^{(i)} \cdots w_n^{(i)}}_{\text{need not be eigen vectors.}} \}$

$\{ w_1^{(n)}, w_2^{(n)} \cdots w_n^{(n)} \}$ $\boxtimes$

$$M = \begin{bmatrix} \lambda_1 & \\ & \tilde{M} \end{bmatrix} \qquad \tilde{M} \in \mathbb{R}^{n-1 \times n-1}$$

$$\tilde{M} v = \lambda_2 v \qquad \underline{v \in \mathbb{R}^{n-1}}$$

Then $\begin{bmatrix} 0 \\ v \end{bmatrix}$ is e. vect of M with eigenvalue λ_2

$$M \begin{bmatrix} 0 \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{M} v \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda_2 v \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ v \end{bmatrix}$$

Spectral Decomposition Theorem

M is symmetric $\Longleftrightarrow$ M can be written as

$$M = \sum_{i=1}^{n} \lambda_i u_i u_i^T$$

where λ_i, u_i are eigenvalue, vectors and u_i's are O.N.B.

$vv^T \in \mathbb{R}^{n\times n}$, symmetric, outer product

rank $= 1$

Proof:

$\Leftarrow$

sum of sym matrice is sym.

$\Rightarrow$

$M \quad \lambda_1', \ldots \lambda_n'$

$u_1' \ldots u_n'$ O.N.B

$$M' = \sum_{i=1}^{n} \lambda_i' u_i' u_i'^T$$

Is $M' = M$?

$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$

$(vv^T)_{ij} = v_i v_j$

Claim

If $M' = \sum_{i=1}^{n} \lambda_i' u_i' u_i'^T$

then λ_i' has to be eigenvalues of M'

u_i' has to be eigenvectors of M'

Proof:

$$M' u_1' = \sum_{i=1}^{n} \lambda_i' u_i' (u_i'^T u_1')$$

$$= \lambda_1' u_1' \cdot 1$$

$\boxtimes$

$\forall v \in \mathbb{R}^n,$

$$Mv = M\left(\sum_{i=1}^{n} \langle v, u_i' \rangle u_i' \right)$$

$$= \sum_{i=1}^{n} \langle v, u_i' \rangle \lambda_i' u_i'$$

$$= M' v \Rightarrow M = M'$$