

Previously.

Spectral Thm.

$M \in \mathbb{R}^{n \times n}$ is symmetric

$\Leftrightarrow M$ has

O.N.E.V. \mathcal{B}
orthonormal
eigenvector
basis.

Spectral Decomposition

$M \in \mathbb{R}^{n \times n}$ is symmetric

$\Leftrightarrow \exists \lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $u_1, \dots, u_n \in \mathbb{R}^n$
s.t. $M = \sum_{i=1}^n \lambda_i u_i u_i^T$

Complex Case

$$M \in \mathbb{C}^{n \times n}$$

$$\langle v, w \rangle = \bar{v}^T w$$

M has real eigen value.

$$\bar{v}^t M v = \bar{v}^t (M v) = \lambda \langle v, v \rangle$$

$$= (\bar{v}^t M) v$$

$$= (M^t \bar{v})^t v$$

$$= (\bar{M}^t \bar{v})^t v \quad (M \text{ is real})$$

$$= \overline{(M v)}^t v \quad (M \text{ is sym})$$

$$= \bar{\lambda} \langle v, v \rangle$$

$$\lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}$$

$$\bar{M}^t = M$$

$$\bar{M}_{ij} = M_{ji}$$

def
(Hermitian
matrix)

$$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

x

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

✓

S.D.F

$M \in \mathbb{C}^{n \times n}$ is Hermitian

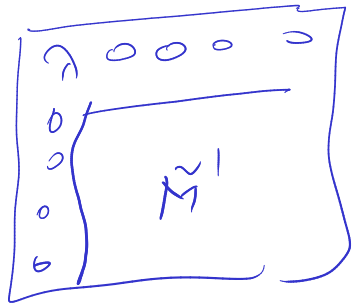
$\Leftrightarrow M$ has ONEB

Proof:

- $\lambda \in \mathbb{R}$
- $v \in \mathbb{C}^n$

- O.N.B $\{w_1, \dots, w_n\}$ $w_1 = v$

$$M' = P^{-1} M P$$



$M^{\sim 1} \in \mathbb{C}^{(n-1) \times (n-1)}$ is Hermitian

$$P^{-1} = \overline{P}^T \text{ (unitary matrices)}$$

\Downarrow

M' is Hermitian

\square

$M \in \mathbb{C}^{n \times n}$ is Hermitian

$\Leftrightarrow \exists \lambda_1, \dots, \lambda_n \in \mathbb{R}, u_1, \dots, u_n \in \mathbb{C}^n$
O.N.E.B

$$M = \sum_{i=1}^n \lambda_i u_i \overline{u_i}^T$$

What can we say about

$$M \in \mathbb{R}^{m \times n} \quad m \geq n$$

Singular Value Decomposition

$$M: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$M^T: \mathbb{R}^m \rightarrow \mathbb{R}^n$$

$$M^T M: \mathbb{R}^n \rightarrow \mathbb{R}^n, \text{ symmetric}$$

$$M M^T: \mathbb{R}^m \rightarrow \mathbb{R}^m, \text{ symmetric}$$

$M^T M$ has O.N.E.B u_1, \dots, u_n
 $\lambda_1, \dots, \lambda_n$

$$\text{rank}(MM^T) \leq \text{rank}(M)$$

\uparrow
 $\mathbb{R}^{m \times m}$

$$\text{rank}(M) = n$$

$\text{rank} = \dim(\text{range})$

$$M^T M u_i = \lambda_i u_i$$

$$\Rightarrow (M(M^T M) u_i) = \lambda_i (M u_i)$$

$\Rightarrow M u_i$ is eigenvector of MM^T
with eigenvalue λ_i

$$M u_1, \dots, M u_n$$

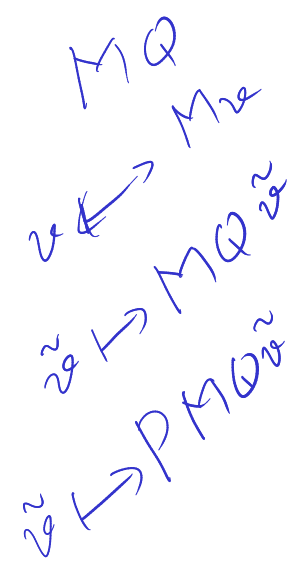
$$\begin{aligned} (M u_i)^T (M u_j) &= u_i^T (M^T M u_j) \\ &= \lambda_j u_i^T u_j \end{aligned}$$

if $i=j$, $\lambda_i = (Mu_i)^T (Mu_i)$
 $\Rightarrow \lambda_i \geq 0$
 if $i \neq j$, 0

$v_i = \frac{Mu_i}{\sqrt{\lambda_i}}, \dots, \frac{Mu_n}{\sqrt{\lambda_n}} = v_n$

$M: \mathbb{R}^n \rightarrow \mathbb{R}^m$

$v_i = Mu_i \quad i=1, \dots, n$



What is the matrix of M
 when basis for \mathbb{R}^n u_1, \dots, u_n
 \mathbb{R}^m $v_1, \dots, v_n, w_{n+1}, \dots, w_m$

$$u_i \rightarrow$$

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tilde{u}_i$$

$$v_i \Rightarrow \begin{bmatrix} 0 \\ \vdots \\ 0 \\ p \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i$$

$$M' = PMQ$$

$$\rightarrow M' \tilde{u}_i = \sqrt{\lambda_i} \tilde{v}_i \quad i=1, \dots, n$$

$$\begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & \sqrt{\lambda_2} & & & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & \sqrt{\lambda_n} \\ \hline 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

$$T: V \rightarrow V$$

$$T: V \rightarrow W$$

$$T \rightarrow M$$

w. n. t. a basis
for V, W

$$\lambda_i \geq 0$$

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \dots \geq \lambda_n$$

$$M = \sum_{i=1}^n \sqrt{\lambda_i} v_i u_i^T$$

S.V.D Theorem

Any $M \in \mathbb{R}^{m \times n}$ singular value

$$M = \sum_{i=1}^r \sqrt{\lambda_i} v_i u_i^T$$

$$r = \text{rank}(M)$$

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0$$

where $\sqrt{\lambda_i} \geq 0$ and

u_1, \dots, u_n O.N.B

v_1, \dots, v_n O.N

Proof:

$$T_M(v) = \sum_{i=1}^r \sqrt{\lambda_i} v_i (\langle \cdot, u_i \rangle v)$$