Previously:

Spectral Thm.

$M \in \mathbb{R}^{n\times n}$ is symmetric

$\iff$ M has O.N.E.V B

orthonormal eigenvector basis.

Spectral Decomposition

$M \in \mathbb{R}^{n\times n}$ is symmetric

$\iff \exists \lambda_1 \ldots \lambda_n \in \mathbb{R}$ and $u_1 \ldots u_n \in \mathbb{R}^n$

s.t. $M = \sum_{i=1}^{n} \lambda_i u_i u_i^T$

Complex Case

$\langle v, w \rangle = \bar{v}^T w$

$M \in \mathbb{C}^{n\times n}$

M has real eigen value.

$\bar{v}^t M v = \bar{v}^t (Mv) = \lambda \langle v, v \rangle$

$= (\bar{v}^t M) v$

$= (M^t \bar{v})^t v$

$\bar{M}^t = M$

$= (\bar{M}^t \bar{v})^t v$ (M is real)

$= (\overline{Mv})^t v$ (M is sym)

$= \bar{\lambda} \langle v, v \rangle$

$\lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}$

def

$\bar{M}^t = M \qquad \overline{M_{ij}} = M_{ji}$ (Hermitian matrices)

$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ ×

$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ ✓

S.D.T

$M \in \mathbb{C}^{n\times n}$ is Hermitian

$\iff M$ has ONEVB

Proof:

- $\lambda \in \mathbb{R}$, $v \in \mathbb{C}^n$
- O.N.B $\{w_1 \dots w_n\}$ $\quad w_1 = v$

$$M' = P^{-1} M P \qquad P^{-1} = \bar{P}^T \text{ (unitary matrices)}$$

$$\Downarrow$$

M' is Hermitian

$$\left[\begin{array}{c|cccc} \lambda & 0 & 0 & 0 & 0 \\ \hline 0 & & & & \\ 0 & & \tilde{M}' & & \\ 0 & & & & \\ 0 & & & & \end{array}\right]$$

$\tilde{M}' \in \mathbb{C}^{n-1\times n-1}$ is Hermitian

$\boxtimes$

$M \in \mathbb{C}^{n \times n}$ is Hermitian

$\Leftrightarrow \exists \lambda_1 \dots \lambda_n \in \mathbb{R}$, $u_1 \dots u_n \in \mathbb{C}^n$

O.N.E.B

$$M = \sum_{i=1}^{n} \lambda_i u_i \overline{u_i}^T$$

What can we say about

$M \in \mathbb{R}^{m \times n}$ $m \geq n$

Singular Value Decomposition

$M : \mathbb{R}^n \to \mathbb{R}^m$

$M^T : \mathbb{R}^m \to \mathbb{R}^n$

$M^T M : \mathbb{R}^n \to \mathbb{R}^n$, symmetric

$M M^T : \mathbb{R}^m \to \mathbb{R}^m$, symmetric

$M^T M$ has O.N.E.B $u_1 \cdots u_n$

$\lambda_1 \cdots \lambda_n$

$\operatorname{rank}(M M^T) \leq \operatorname{rank}(M)$

$\mathbb{R}^{m \times m}$

$\operatorname{rank}(M) = n$

$\boxed{\text{rank} = \dim(\text{range})}$

$$M^T M u_1 = \lambda_1 u_1$$

$$\Rightarrow (M(M^T)(Mu_1)) = \lambda_1 (Mu_1)$$

$\Rightarrow Mu_i$ is eigenvector of MM^T

with eigenvalue λ_i

$Mu_1, \ldots, Mu_n$

$$(Mu_i)^T (Mu_j) = u_i^T (M^T M u_j)$$

$$= \lambda_j \, u_i^T u_j$$

if $i = j$, $\quad \lambda_i = (Mu_i)^T (Mu_i)$

$\Rightarrow \lambda_i \geq 0$

if $i \neq j$, $\quad 0$

$$v_1 = \frac{Mu_1}{\sqrt{\lambda_1}}, \ \dots \ , \ \frac{Mu_n}{\sqrt{\lambda_n}} = v_n$$

$$M : \underset{V}{\mathbb{R}^n} \longrightarrow \underset{W}{\mathbb{R}^m}$$

$v_i = Mu_i \quad i = 1, \dots n$

$v_1 \dots v_n \underbrace{w_{n+1} \dots w_m}$ O.N.B for $\mathbb{R}^m$

What is the matrix of M

when basis for $\mathbb{R}^n$ $\quad u_1 \dots u_n$

$\mathbb{R}^m \quad v_1 \dots v_n \, w_{n+1} \dots w_m$

MQ

$v \mapsto Mv$

$\tilde{v} \mapsto MQ\tilde{v}$

$\tilde{v} \mapsto PMQ\tilde{v}$

$$u_i \longrightarrow \tilde{u}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad v_i \Longrightarrow \tilde{v}_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow i$$

$$M' = PMQ$$

$$\Rightarrow \quad M'\tilde{u}_i = \sqrt{\lambda_i}\,\tilde{v}_i \qquad i = 1, \dots, n$$

$$\begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sqrt{\lambda_n} \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$T : V \to V$

$T : V \to W$

$T \longrightarrow M$

w.r.t a basis for V, W

$\lambda_i \geq 0$

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots \geq \lambda_n$$

$$M = \sum_{i=1}^{n} \sqrt{\lambda_i}\, v_i\, u_i^T$$

S.V.D Theorem

Any $M \in \mathbb{R}^{m \times n}$

singular value

$r = \text{rank}(M)$

$$M = \sum_{i=1}^{r} \sqrt{\lambda_i}\, v_i\, u_i^T$$

$\lambda_1 \geq \lambda_2 \cdots \lambda_r = \geq 0$

where $\sqrt{\lambda_i} \geq 0$ and

$u_1 \ldots u_n$ O.N.B

$v_1 \ldots v_n$ O.N

Proof:

$$\boxed{T_M(v) = \sum_{i=1}^{n} \sqrt{\lambda_i}\, v_i \left(\langle u_i, v\rangle\right)}$$