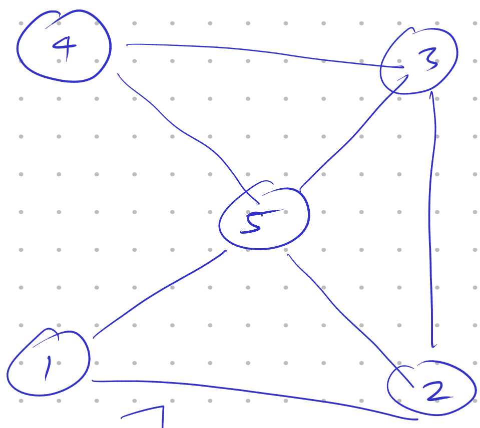
# Linear Algebra Part-II Lec 2

$$p^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$ (initial distribution)




$$p^1 = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}$$

$$p^2 = \begin{bmatrix} \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{4} \\ \frac{1}{2} \cdot \frac{1}{4} \\ \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{3} \\ \\ \end{bmatrix}$$

$$p^3 = M p^2$$

$$\boxed{p^t = M p^{t-1}}$$

$$M = \begin{bmatrix} 0 & 1/3 & 0 & 0 & 1/4 \\ 1/2 & 0 & 1/3 & 0 & 1/4 \\ 0 & 1/3 & 0 & 1/2 & 1/4 \\ 0 & 0 & 1/3 & 0 & 1/4 \\ 1/2 & 1/3 & 1/3 & 1/2 & 0 \end{bmatrix}$$

$i$th column of $M$ is the distribution after 1 step starting from node $i$

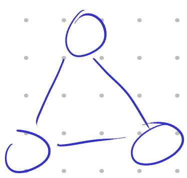
What happens after a long time?

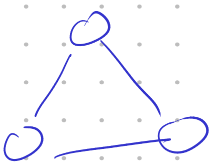
$p^0, p^1, p^2 \dots \quad p^t \dots\dots$

as $t \to \infty$, $p^t \longrightarrow p^*$ ?

assuming $p^t$ converges (equilibrium)

$$\boxed{M p^* = p^*}$$



$$\begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \\ 4 \end{bmatrix}$$

$$p^0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$p^0 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \qquad p^0 = \begin{bmatrix} 99/100 \\ 1/100 \end{bmatrix}$$

# Eigen Vectors and Eigenvalues

$M: \mathbb{F}^n \to \mathbb{F}^n$ (linear operator)

**Def:** $v \neq 0$ is said to be an eigenvector of $M$ iff $\exists\, \lambda \neq 0 \in \mathbb{F}$ s.t.

$M \in \mathbb{F}^{n \times n}$

$\lambda \in \mathbb{F}$

$$M \cdot v = \lambda \cdot v$$

and $\lambda$ is eigenvalue of $v$.

$\Rightarrow$ equilibrium has eigenvalue $= 1$

$$M^t v = \lambda^t \cdot v$$

- $Mv = \lambda v$
  $w = \alpha v$

$\Rightarrow$ $w$ is also eigenvector with eigenvalue $\lambda$

# How to find eigen vectors?

Suppose you are given $\lambda$,

$Mv = \lambda v$ solve lin eqn.

How to find $\lambda$?

$$Mv - \lambda v = 0 \Rightarrow \overbrace{(M-\lambda I)}^{\in \mathbb{F}^{n\times n}} v = 0$$

$$\Downarrow \quad \neq 0$$

$$v \in \ker(M-\lambda I)$$

$$\Downarrow$$

$$\det(M-\lambda I) = 0$$

$$\Downarrow$$

polynomial with $\lambda$ variable of degree $\leq n$

$$\begin{bmatrix} M_{11}-\lambda & & M_{ij} \\ & M_{22}-\lambda & \\ M_{ij} & & M_{33}-\lambda \end{bmatrix}$$

roots of the polynomials are eigen values.

# distinct e. vals $\leq n$.

$\lambda_1 \ldots \lambda_n$

Find a matrix $M$ with eigen vals $\lambda_1 \ldots \lambda_n$.

$$\begin{bmatrix} \lambda_1 & & & 0 \\ & \ddots & & \\ & & \lambda_i & \\ 0 & & & \ddots \\ & & & & \lambda_n \end{bmatrix} \qquad e_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow$$

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Suppose you have eigen vectors $v_1 \ldots v_k$ with eigen values $\underbrace{\lambda_1 \ldots \lambda_k}_{\text{distinct}}$ respectively.

Can $v_1 \ldots v_k$ be linearly dependent?

$\ker(M) = \{ v \in V : Mv = 0 \}$

Suppose $v_1 \ldots v_k$ are lin. dependent.

Let $r$ be the smallest no. s.t.

$v_1 \ldots v_r$ is lin. dependent.

but $v_1 \ldots v_{r-1}$ are independent.

$$v_r = \sum_{i=1}^{r-1} \alpha_i v_i \quad \text{——} \quad 1.$$

$$\lambda_r v_r = \sum_{i=1}^{r-1} \alpha_i \lambda_i v_i \quad \text{——} \quad 2.$$

$$\sum_{i=1}^{r-1} \alpha_i (\lambda_i - \lambda_r) v_i = 0$$

$v_1 \ldots \underline{v_{r-1}}$ are linear dependent

Hence $v_1 \ldots v_r$ are linearly independent.