- Eigen values and eigen vectors

- Eigen. vector corresponding to distinct eigen values are linearly independent. ✓

- # eigen value $\leq n$

$$\det(M - \lambda I) = 0$$

$M \in \mathbb{R}^{n \times n}$

→ poly over real field # $\leq n$

→ poly over $\mathbb{C}$ field # soln $= n$

Suppose M had n linearly independent eigen vectors

$\{v_1 \ldots v_n\}$ $\lambda_1 \ldots \lambda_n$

$$e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow i$$

$$w = \alpha_1 v_1 + \alpha_2 v_2 \ldots + \alpha_n v_n$$

$$Mw = \lambda_1 \alpha_1 v_1 + \lambda_2 \alpha_2 v_2 \ldots + \lambda_n \alpha_n v_n$$

What is M in $\{v_1 \ldots v_n\}$ basis?

Mw

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ldots & \\ 0 & & \lambda_n \end{bmatrix} = P^{-1} M P$$

where P is an invertible matrix

$$T: V \to V$$

$$T: V \to W$$

$$B = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \ldots & v_n \\ | & | & & | \end{bmatrix}$$

Mw (original std basis)

$P^{-1}MP\bar{w}$ (eigenvector basis)

$\bar{w}$: coordinates of w in new basis

P: converts $\bar{w} \to w$

P^{-1}: convert old basis coordinates to new coordinates.

Hence $P^{-1}MP$ is matrix in new basis

$$
\begin{tikzcd}
 & M \\
T \arrow[ur, "old"] \arrow[dr, "new"'] & \\
 & M'
\end{tikzcd}
$$

$$\boxed{M' = P^{-1}MP}$$

P invertible change of basis matrix

If P correspond to change of basis to eigen vector basis, then M' is a diagonal matrix

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What happens when there are no n linearly indep eigenvectors?

<u>Equivalence relation</u>

$M \equiv M'$ iff $\exists$ invertible P $M' = P^{-1} M P$

If $M \equiv$ to a diagonal matrix, then M is <u>diagonalizable</u>

Are there matrices that are not diagonalizable?

$$M'' = Q^{-1} M' Q$$

$$M'' = Q^{-1} P^{-1} M P Q$$

eigen value can be 0

If M is non invertible then
M has eigen value of 0

$\underbrace{v_1, v_2, v_3}_{\substack{\uparrow \\ \text{null space}}}$ $\quad v_n$

$$\begin{bmatrix} 0 & & & \\ & 0 & & 0 \\ & & \lambda_3 & \\ & 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}$$

$\det(M - \lambda I) = 0$ → characteristic polynomial of M

over $\mathbb{C}$ field

$$\prod_{i=1}^{n} (\lambda - \alpha_i) = 0$$

$\underbrace{\alpha_1 = \alpha_2 = \alpha_3}_{=a}$ $\quad \underbrace{\alpha_4 = \alpha_5}_{=b}$ $\quad \underbrace{\alpha_6 = \alpha_7 = \alpha_8}_{=c}$

# of term with most α is called algebraic multiplicity

geometric multiplicity of a = maximal # of linearly independent eigenvectors with eigenvalue a

$n = 8$

$$(x-a)^3 (x-b)^2 (x-c)^3 = 0$$

linearly independent eigenvectors with eigenvalue $a = 3$

" $b = 2$

" $c = 3$

If algebraic multiplicity = geometric $\longrightarrow$ then matrix is diagonalizable

But these are matrices s.t. alg $\neq$ geo multiplicity.

So all matrices are not diagonalizable

$$\text{Eigen Space}_M(\lambda) = \{v \in V : Mv = \lambda v\}$$

Is it a subspace?

v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_8