# Orthogonal Vector

$v, w \in V$ (vector space)
are orthogonal iff $\langle v, w \rangle = 0$.

**Claim:** $v_1 \dots v_n \in V$ ($n$-dim vector space)

$v_i \neq 0$ and are orthogonal
$\forall i \neq j \; \langle v_i, v_j \rangle = 0$

Then $v_1 \dots v_n$ forms a basis of $V$

**Proof:**

Suppose $v_1, \dots v_n$ linearly dependent (not all $\alpha_i = 0$)

$$v_m = \sum_{i=1}^{m-1} \alpha_i v_i$$

$$\langle v_m, v_1 \rangle = \left\langle \sum_{i=1}^{m-1} \alpha_i v_i, v_1 \right\rangle$$

$$\| \qquad\qquad\qquad = \sum_{i=1}^{m-1} \alpha_i \langle v_i, v_1 \rangle$$

$$0$$

$$= \alpha_1 \langle v_1, v_1 \rangle + \underbrace{\sum_{i=2}^{n-2} \alpha_i \langle v_i, v_1 \rangle}_{=0}$$

$$\Rightarrow \langle v_1, v_1 \rangle = 0$$

$$\Rightarrow \boxed{v_1 = 0} \quad \text{(contradiction)}$$

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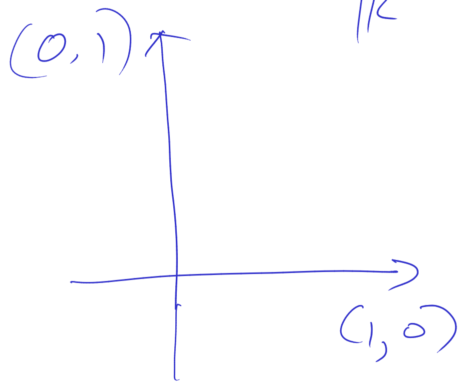
Suppose $v_1 \dots v_n$ are orthogonal

$\mathbb{R}^2$




$$\|v_1 + v_2 \dots v_n\|^2$$

$$= \|v_1\|^2 + \|v_2\|^2 + \dots + \|v_n\|^2$$

(Pythagoras Theorem)

$$\|v\| := \sqrt{\langle v, v \rangle}$$

$$\|v_1 + \dots + v_n\|^2$$

$$= \langle \quad , \quad \rangle$$

$$=$$

$$\|v_1 + v_2\|$$

$$= \langle v_1 + v_2, v_1 + v_2 \rangle$$

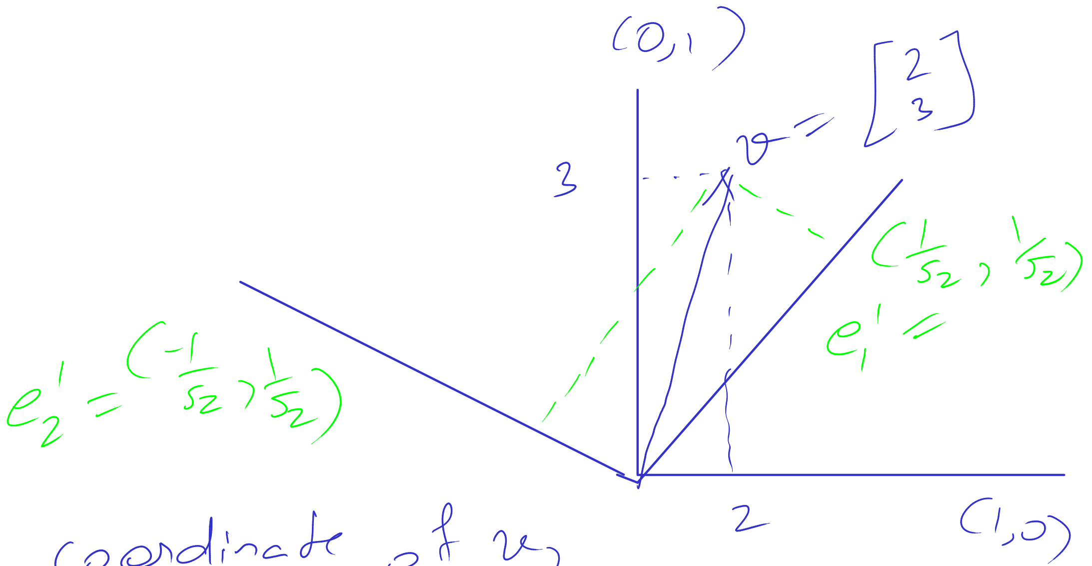
$$= \langle v_1, v_1 \rangle + \langle v_2, v_2 \rangle + \cancel{\langle v_1, v_2 \rangle}\ 0 + \langle v_2, v_1 \rangle\ 0$$

# Orthonormal Vectors

$v_1, \ldots v_n$ are orthonormal

if - $\forall i \; \langle v_i \cdot v_i \rangle = 1$

- $v_1 \ldots v_n$ are orthogonal




Std basis coordinate of $v$,

is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Coordinates of $v$ wrt to $e_1', e_2'$

$$\begin{bmatrix} \langle v, e_1' \rangle \\ \\ \langle v, e_2' \rangle \end{bmatrix}$$

Bilinear forms $\langle v, w \rangle = v^T A w$ ↳ full rank

Claim: If $v_1 \dots v_n$ orthonormal basis then any $w \in V$

$$w = \langle w, v_1 \rangle v_1 + \langle w, v_2 \rangle v_2 + \dots + \langle w, v_n \rangle v_n$$

$$\begin{bmatrix} \langle w, v_1 \rangle \\ \langle w, v_2 \rangle \\ \vdots \\ \langle w, v_n \rangle \end{bmatrix}$$

coordinate w.r.t $v_1 \dots v_n$

Proof:

$$w = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

$$\langle w, v_1 \rangle = \alpha_1 \langle v_1, v_1 \rangle + \alpha_2 \langle v_2, v_1 \rangle + \dots + \alpha_n \langle v_n, v_1 \rangle$$

$$\alpha_1 = \langle w, v_1 \rangle$$

$v_1 \,.\, v_2 \,..\, v_n$ are orthogonal.

$$w = \frac{\langle w, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle w, v_2 \rangle}{\|v_2\|^2} v_2$$

$$\dots + \frac{\langle w, v_n \rangle}{\|v_n\|^2} v_n$$

How do you find an orthonormal basis?

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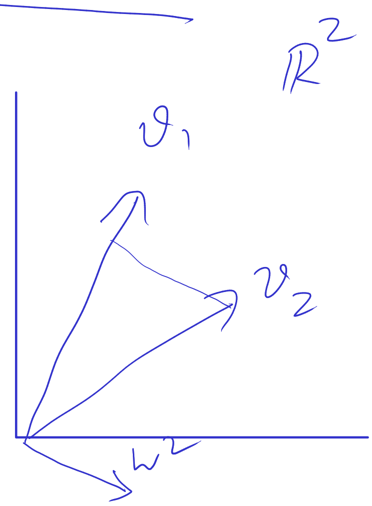
Let $v_1 \dots v_n$ be a basis




$w_1 = v_1$

$$w_2 := v_2 - \underbrace{\left( \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} \right)}_{\text{scalar}} w_1$$

$w_1 \perp w_2$ (mean $w_1$ is orthogonal to $w_2$)

$$\langle w_2, w_1 \rangle = \langle v_2, w_1 \rangle - \frac{\langle v_2, w_1 \rangle}{\langle w_1, \cancel{w_1} \rangle} \langle \cancel{w_1}, w_1 \rangle$$

$$= 0$$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} w_1$$

$$- \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} w_2$$

$w_1 \perp w_2 \perp w_3$

$w_1 \perp w_3$ $\quad$ $w_2 \perp w_3$

CB terms

$$\langle w_2, w_3 \rangle = \langle w_2, v_3 \rangle$$

$$- \frac{\langle v_3, w_2 \rangle}{\cancel{\|w_2\|^2}} \cancel{\langle w_2, w_2 \rangle}$$

$=$ $\quad$ $\vdash$

$$\langle w_3, w_2 \rangle = \langle v_3, w_2 \rangle \quad = 0$$

$$- \frac{\langle v_3, w_2 \rangle \cancel{\langle w_2, w_2 \rangle}}{\cancel{\|w_2\|^2}}$$

$w_4$

$\vdots$

Gram schmidt orthogonalization

$w_n$

$w_1, \dots w_n$ are orthogonal.

$\downarrow$

$\frac{w_1}{\|w_1\|} \dots \frac{w_n}{\|w_n\|}$ are orthonormal.