Gram-Schmidt Orthogonalization

$$\mathrm{proj}_w(v) = \underbrace{\frac{\langle v, w\rangle}{\langle w, w\rangle}}_{\text{scalar}} w \quad \text{— vector}$$

$v_1 \dots v_n$ a basis for vector space

$\downarrow$

$w_1 \dots w_n$ orthogonal basis

$w_1 = v_1$ $\qquad w_1 \perp w_2$

$w_2 = v_2 - \mathrm{proj}_{w_1}(v_2)$

$w_3 = v_3 - \mathrm{proj}_{w_2}(v_3) - \mathrm{proj}_{w_1}(v_3)$

$\vdots$

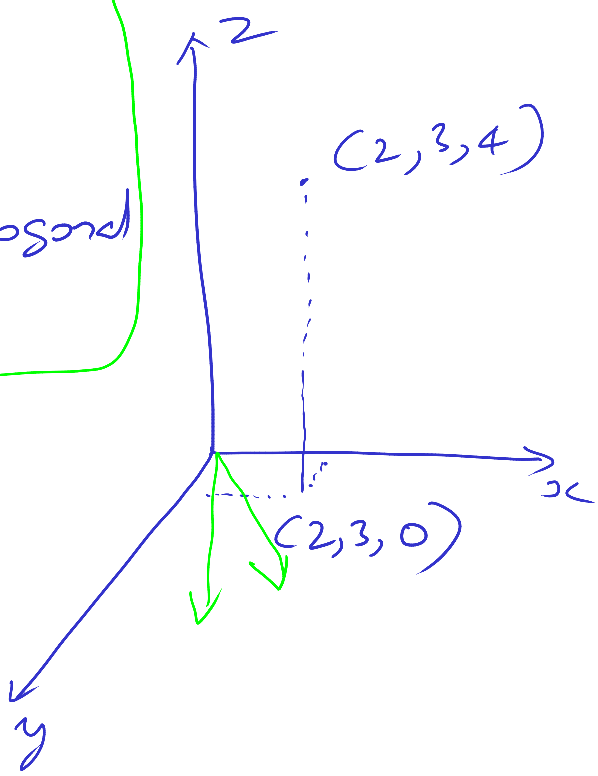
$$w_n = v_n - \sum_{i=1}^{n-1} \mathrm{proj}_{w_i}(v_n)$$

Orthogonality for Subspaces

$v \perp \mathcal{U}$ means $\forall u \in \mathcal{U},\ v \perp u$

$$\operatorname{proj}_{\mathcal{U}}(v) = \sum_{i=1}^{k} \operatorname{proj}_{u_i}(v)$$

where $u_1 \dots u_k$ are orthogonal basis for $\mathcal{U}$.



$$\cancel{v = \sum_{i=1}^{n} \frac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle} v_i}$$

Orthogonal Complement

Let $\mathcal{U}$ be a subset V (n-dim vector space)

$$\mathcal{U}^{\perp} = \{ v \in V : \forall u \in \mathcal{U},\ \langle v, u \rangle = 0 \}$$

$U^\perp$ a subspace?

$a, b \in U^\perp$

$\Downarrow$

$\forall u \in U, \ \langle a, u\rangle = 0 \qquad \Rightarrow \forall u \in U,$

$\langle b, u\rangle = 0 \qquad\qquad \langle a+b, u\rangle = 0$

$\Rightarrow a+b \in U^\perp$

Can $v \in U \cap U^\perp$?

$\Downarrow$

$\langle v, v\rangle = 0 \Rightarrow v = 0$

Suppose U is a subspace of V (n dim)

<u>Claim:</u> $V = U \oplus U^\perp$

<u>Proof:</u> $U \cap U^\perp = \{0\}$

$\Rightarrow V \supseteq U \oplus U^\perp$

assume $v \in V \setminus U \oplus U^\perp$

$v \not\perp \mathcal{U}$

$$v = \operatorname{proj}_{\mathcal{U}}(v) + \Big(\underbrace{v - \operatorname{proj}_{\mathcal{U}}(v)}_{=\,w}\Big)$$

$w \perp \mathcal{U}$?

$\operatorname{proj}_{\mathcal{U}}(v) \in \mathcal{U}$

$v - \operatorname{proj}_{\mathcal{U}}(v) \in \mathcal{U}^{\perp}$

(contradiction)

$M \in \mathbb{R}^{n \times n}$

$$\operatorname{kernel}(M) = \{ v \in V : Mv = 0 \}$$

$$\begin{bmatrix} & & \end{bmatrix} \begin{bmatrix} v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad = \operatorname{span}(\operatorname{rows}(M))^{\perp}$$

$$= \operatorname{rows}(M)^{\perp}$$

$S^\perp = \text{span}(S)^\perp$ $\quad$ S is any subset

$S^\perp \subseteq \text{span}(S)^\perp$ ✓

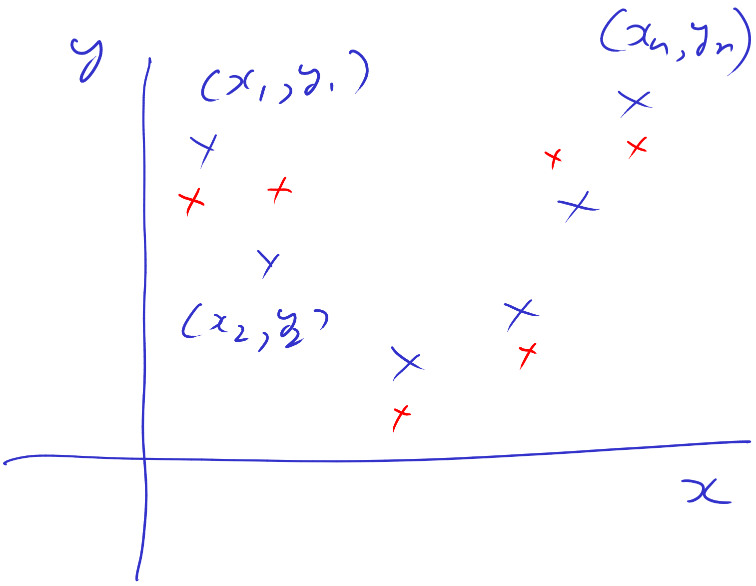
$\supseteq$ ✓

$v \in \text{span}(S)^\perp \Rightarrow v \in S^\perp$

Polynomial Fitting

$ax^2+bx+c=y$

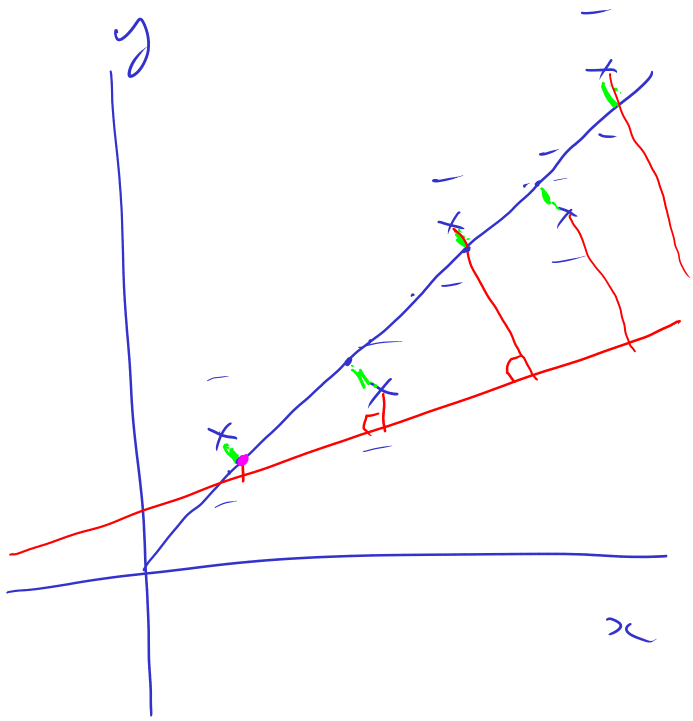
$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ & & \\ & & \\ x_n^2 & x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$



Fitting with errors

x_1, y_1

x_n, y_n



We want to find a line which minimizes the sum of projections to the lines

$y = mx + c$

$$\underbrace{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}}_{M} \begin{bmatrix} m \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}}_{=y}$$

doesn't have a solution

$\begin{bmatrix} y_1 \\ y_n \end{bmatrix} \notin$ colum span(M)

Let $\hat{y} = \mathrm{proj}_{\mathrm{colum}(M)}(y)$