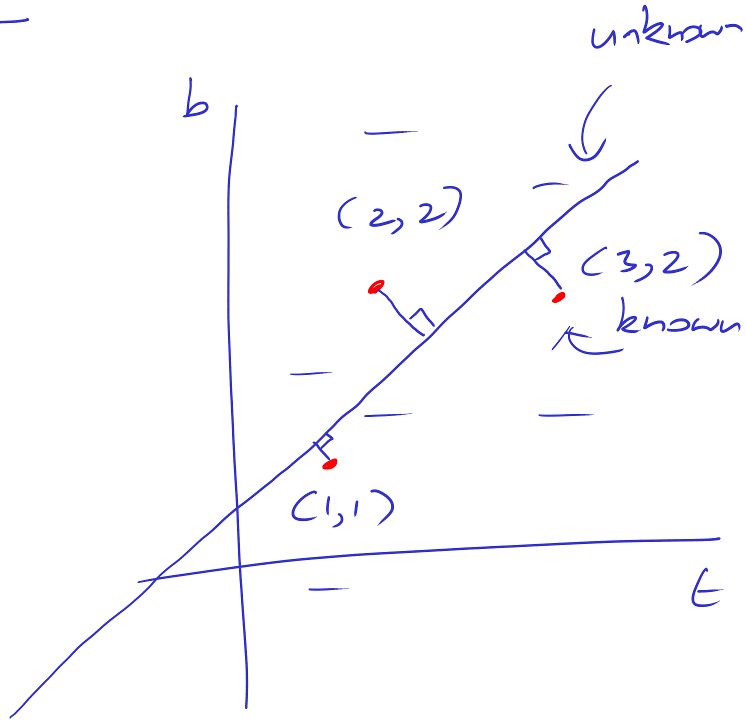
Fitting with Errors

$b = C + Dt$

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$



Consider Range(A)

$$\text{Range}(A) = \text{Col Span}(A)$$

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \notin \underbrace{\text{Col Span}(A)}_{= W}$$

Can we find a vector v, which $\in W$ and $\left\| v - \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\|_2$ is minized?

$\mathrm{proj}_W(v)$ minimizes.

<u>Claim</u>

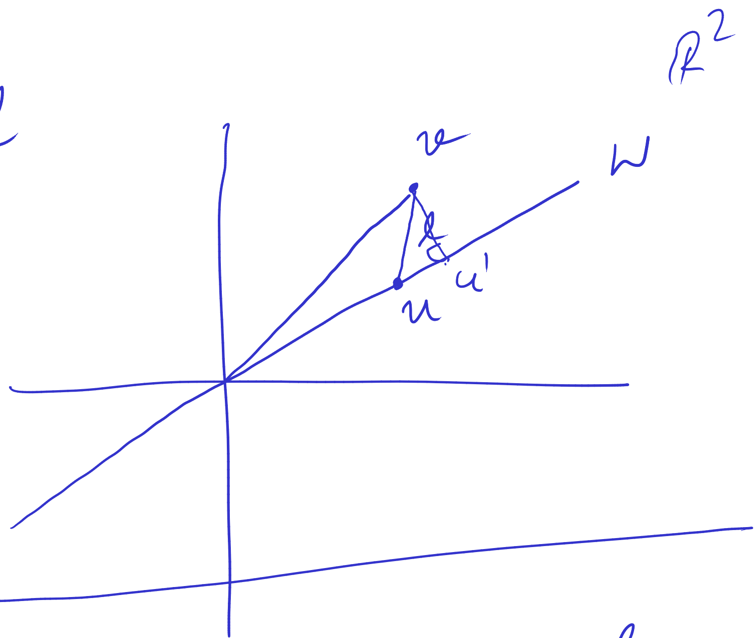
$$\mathrm{proj}_W(v) = \underset{u \in W}{\arg\min} \underbrace{\|u - v\|_2}_{f(u)}$$

Find $u \in W$ that minimizes $\|u-v\|_2$

<u>Proof:</u>

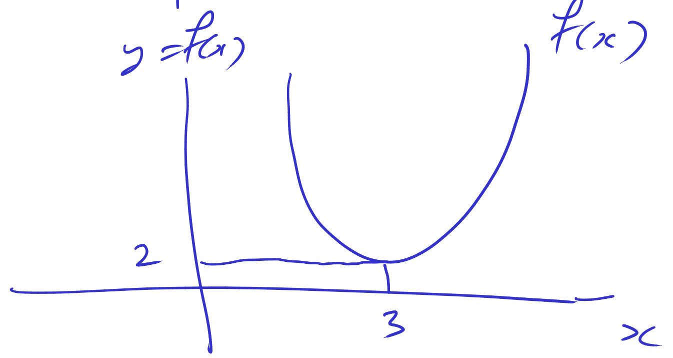
We want to minimize ℓ

$u' = \mathrm{proj}_W(v)$



$$\min_x f(x) = 2$$

$$\underset{x}{\arg\min}\, f(x) = 3$$



$Ax = b$ has no solution

$b \notin \text{Col Span}(A) = W$

$\boxed{Ax = \text{proj}_W(b)} \iff \boxed{A^T A x = A^T b}$

Goal: To find $\hat{x}$ s.t.

- $\|A\hat{x} - b\|^2$ is minimized.

$$b = \underbrace{\text{proj}_W(b)}_{= \hat{b}} + \underbrace{\text{proj}_{W^\perp}(b)}_{b^\perp}$$

- $\min_{\hat{x}} \| \underbrace{A\hat{x} - \hat{b}}_{\in W} - \underbrace{b^\perp}_{\in W^\perp} \|^2$

- $\min_{\hat{x}} \left(\underbrace{\|A\hat{x} - \hat{b}\|^2}_{= 0} + \|b^\perp\|^2 \right)$

$\Rightarrow b = A\hat{x} + b^\perp$

$b^\perp = A\hat{x} - b$

Solve

$$A\hat{x} = \hat{b}$$

where $\hat{b} = \text{proj}_W (b)$

$W = \text{Col space}(A)$

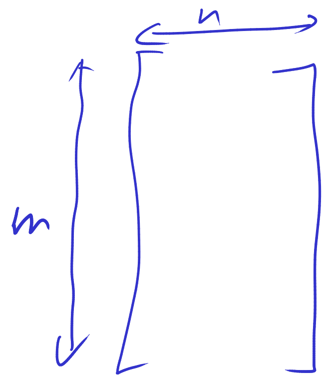
"assume column(A) are linearly independent

then there is a unique solution."

<u>Theorem:</u> $A \in \mathbb{R}^{m\times n}$ is a matrix with $m > n$

linearly independent columns

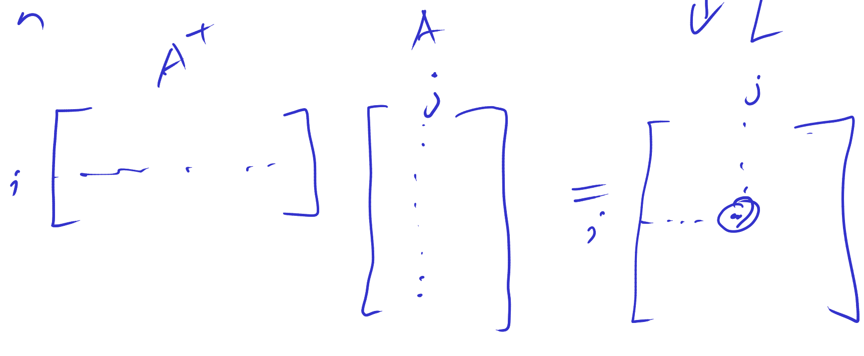
show that $A^T A$ is invertible



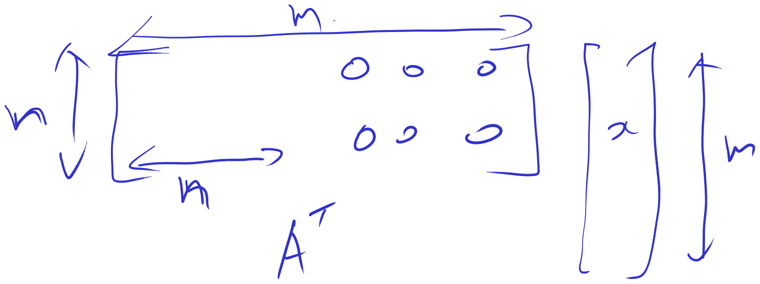
<u>Proof:</u>

$A^T A \in \mathbb{R}^{n\times n}$

$Ax = 0 \Rightarrow x = 0$



$A^T x = 0 \Rightarrow x = 0$ $\times$



$\dim(V) = m$

$\dim(U) = n$

$A : U \to V$

A^T

Suppose $A^T A$ is not full rank

$\exists x \neq 0,$

$$A^T A x = 0$$

$$x^T A^T A x = 0$$

$$(Ax)^T (Ax) = 0$$

$x \neq 0 \Rightarrow Ax \neq 0$

$\Rightarrow Ax = 0$

$\Rightarrow$ Colm (A) are not lin indep.

(contradiction)

$\square$

$Ax = b \qquad b \notin \text{Col Span}(A)$

$A^T A x = A^T b$