4.1 Fixed Points

Let M be a matrix given by

$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

Given any vector $v(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$, we can create an infinite sequence of vectors $v(1), v(2), \cdots$ by using the rule

$$v(t+1) = Mv(t) \qquad \text{for all natural numbers } t$$

a.) Find all vectors $v(0)$ such that

$$v(0) = v(1) = v(2) = v(3) = \cdots$$

b.) Find all vectors $v(0)$, such that $v(0), v(1), v(2), v(3), \cdots$, belongs to a 1 dimensional subspace.

a.) v(0) has to be eigen vector with eigen value 1

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$V(1) = V(0) \iff M\,V(0) = V(0)$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and any scalar multiples.

b.) M has 1 $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} t \\ -t \end{bmatrix}$

5 $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} t \\ t \end{bmatrix}$

4.2 Commuting Matrices

Simultaneously Diagonalizable.

Let A, B be commuting matrices of dimension $n \times n$ (ie $AB = BA$) and suppose A is diagonalizable with n distinct eigenvalues.

a.) Show that of v is an eigenvector of A with eigenvalue λ, then Bv is also an eigenvector of A with eigenvalue λ.

b.) Show that if v is an eigenvector of A, the v is also an eigenvector of B. Should the eigenvalues be the same?

c.) Explain why the above implies that there is a single change of basis such that A, B are both diagonal in the same basis.

c.) $A = P^{-1} D P$

$B = P^{-1} D' P$ $\Longleftrightarrow$ $AB \overset{\shortparallel}{} BA$

$\Longleftarrow$

$\xrightarrow{?}$

a.)

$$Av = \lambda v$$

$$BAv = \lambda Bv$$

$$A(Bv) = \lambda (Bv) \implies Bv \text{ is eigenvector with eigenvalue } \lambda.$$

b.)

$$\text{Eigen Space}(\lambda) := \{v : Av = \lambda v\}$$

e.vect. corresponding to different eigenvalues are lin. indep $\implies \dim(\text{EigenSpace}(\lambda)) = 1$

$$\implies Bv, v \in \text{Eigen Space}(\lambda)$$

$$Bv = \alpha v$$

$\implies v$ is eigen vector of B with eigenvalue α.

α may not be $=$ to λ.

3

4.3 Decomposition

a.) Let Q be an $n \times n$ orthonormal matrix (columns form an orthonormal basis). For any vectors $v, u \in \mathbb{R}^n$, show that $= \sum u_i v_i$

$$u \cdot v = (Qu) \cdot (Qv) \qquad \text{(dot product)}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

b.) Let $\{u_1, \cdots u_n\}$ be column vectors $\in \mathbb{R}^n$ (ie $n \times 1$ dimensional) that are orthonormal. Suppose $M \in \mathbb{R}^{n \times n}$ ($n \times n$ dimensional matrix) defined by:

$$M = \sum_{i=1}^{n} \alpha_i u_i u_i^T \quad \text{where } \alpha_i \text{ are scalars.}$$

$= u_1 e_1 + u_2 e_2 + u_n e_n$

Note that $u_i u_i^T$ are $n \times n$ dimensional. What are the eigenvectors and eigenvalues of M? (need to explain why)

$$e_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow i$$

a.) $u \cdot v = u^T v$

$$(Qu) \cdot (Qv) = (Qu)^T (Qv)$$
$$= u^T Q^T Q v$$
$$= u^T I v$$
$$= u^T v$$

b.) $u^T u \in \mathbb{R}$ (dot product)

$u u^T \in \mathbb{R}^{n \times n}$ (outer product)

$\operatorname{rank}(u u^T) = 1$

$$\begin{array}{cc} & j \\ i & \begin{bmatrix} & \\ & u_i u_j \\ & \end{bmatrix} \end{array}$$

$$\begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix}$$

$$Mu_1 = \sum_{i=1}^{n} \alpha_i u_i u_i^T u_1$$

$P =$ convert
$e_1 \ldots e_n$
$\downarrow$
$u_1 \quad u_n$

$$= \alpha_1 u_1 (u_1^T u_1) + \sum_{i=2}^{n} \alpha_i u_i \underbrace{(u_i^T u_1)}_{0}$$

$$= \alpha_1 u_1 \quad \left(\begin{bmatrix} \\ \\ \end{bmatrix} \overset{1 \times n}{\begin{bmatrix} & \end{bmatrix}} \overset{n \times 1}{\begin{bmatrix} \\ \\ \end{bmatrix}} \right)$$

$$\begin{bmatrix} \\ u_1 \\ \\ \end{bmatrix} [a] = a u_1$$

Is M symmetric?

$\forall i$, u_i is an eigen vector with eigen value α_i