Spectral Decomposition

Symmetric Matrix (Real)

$M \in \mathbb{R}^{n \times n}$, $M^T = M$ ie $M_{ij} = M_{ji}$

- adjacency matrix of undirected graph

Theorem: A symmetric matrix has real eigen values.

Proof:

$$Mv = \lambda v \rightarrow [M][v] = \lambda [v]$$

$$[v^T]\,[M^T] = \lambda v^T$$

v is a right eigenvector of M with eigenvalue λ

$\Leftrightarrow$ v^T is a left " of M " " "

(for a symmetric matrix)

λ, v can be $\mathbb{C}$

$$\bar{v}^T M v = \bar{v}^T (M v)$$

$$= \bar{v}^T \lambda v$$

$$= \lambda \bar{v}^T v = \lambda \langle v, v \rangle \quad (1)$$

$$= (\bar{v}^T M) v \quad (2)$$

$$= (M \bar{v})^T v = \bar{\lambda} \bar{v}^T \cdot v = \bar{\lambda} \langle v, v \rangle$$

$Mv = \lambda v \Leftrightarrow M\bar{v} = \bar{\lambda}\bar{v}$ (for $M \in \mathbb{R}^{n \times n}$)

$\overline{a+b} = \bar{a} + \bar{b}$

$\overline{ab} = \bar{a} \cdot \bar{b}$

$\lambda = \bar{\lambda}$ (from (1), (2))

$\Rightarrow \lambda \in \mathbb{R}$ $\boxtimes$

Theorem: If v, w has distinct eigen values λ_1, λ_2 then v, w are orthogonal for a symmetric matrices.

$$\bar{v}^T \cdot w = 0$$

Proof:

$$Mv = \lambda_1 v \qquad Mw = \lambda_2 w$$

$$\bar{w}^T M v = \bar{w}^T (Mv)$$

$$= \lambda_1 \bar{w}^T v \quad \text{—} \; 1.$$

$$= (\bar{w}^T M) v$$

$$= \bar{\lambda}_2 \bar{w}^T v = \lambda_2 \bar{w}^T v \quad \text{—} \; 2.$$

$$\underset{\neq 0}{(\lambda_1 - \lambda_2)} \langle w, v \rangle = 0$$

$$\Rightarrow \langle w, v \rangle = 0$$

$\boxtimes$

Thm
For Symmetric Matrices,
~~if there are n distinct eigenvalues~~
iff there is a basis of ortho-normal eigen vectors

Proof:

$\Leftarrow$:

Suppose there is a basis of o.n. eigen vectors (of M)

ie M is diagonalizable.

$$P^{-1} M P = D \qquad P^T P$$

$$P = \begin{bmatrix} | & | & | & | & | \\ | & | & | & | & | \end{bmatrix}$$ columns of P are orthonormal.

$$P^{-1} = P^T \quad (P^T P = I)$$

$$P^T M P = D$$

$\Rightarrow \quad M = P D P^T$

$(D^T = D$ diagonal matrix$)$

$$M^T = (P D P^T)^T = P D P^T$$

$\Rightarrow$ M is symmetric

$\Leftarrow$

$M^T = M$

det of eigen value

$\exists v \neq 0$ s.t $Mv = \lambda v$

Any matrix has atleast 1 eigen values.

$\Rightarrow \quad \exists \overset{\neq 0}{v}, \lambda$ s.t $\quad Mv = \lambda v$

(but geo. mult $= 1$ and alg. mult. $= n$)

$w_1 \ldots \quad w_n$ o.n basis for $\mathbb{R}^n$

s.t $w_1 = v$

$$P^{-1}MP = P^{T}MP$$

$$= \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$