No Free Lunch Theorem

- The hypothesis class of all functions is not PAC Learnable.

<u>Theorem:</u> Let A be any learning algorithm (binary classification)

domain: $\mathcal{X}$ $\quad \mathcal{Y} = \{0,1\}$

# of samples $\leq |\mathcal{X}|/2$

Then there exists f (a function) and a distribution D on $\mathcal{X} \times \{0,1\}$ such that

- $L_D(f) = 0$
- $\Pr_{S \sim D^m} [L_D(A(S)) \geq 1/8] \geq 1/7$

fn outputted by A on S

<u>Proof</u>

$|\mathcal{X}| = 2m$

Alg A sees m samples

# of fns $\mathcal{X} \to \{0,1\} = 2^{2m} = T$

$f_1 \quad f_2 \quad \dots \quad f_T \quad$ fns

$D_1 \quad D_2 \quad \dots \quad D_T \quad$ dist. $\mathcal{X} \times \{0,1\}$

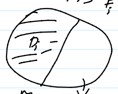
$$D_i((x,y)) = \begin{cases} 1/2m & \text{if } f_i(x) = y \\ 0 & \text{otherwise} \end{cases}$$

Will Prove:

$$\max_{i \in 1..T} \mathbb{E}_{S \sim D_i^m} [L_{D_i}(A(S))] \geq 1/4$$

Markovs Inequality

Marginal of D_i on $\mathcal{X}$ is uniform



$S = (x_1, x_2 \dots x_m)$

# of possible S's $(2m)^m = k$

$S_1 \quad S_2 \quad \dots \quad S_k$

$S_j^i = \{ (x_1, f_i(x_1)), \dots, (x_m, f_i(x_m)) \}$

$$\mathbb{E}_{S \sim D_i^m} L_{D_i}(A(S)) = \frac{1}{k} \sum_{j=1}^{k} L_{D_i}(A(S_j^i))$$

$$\max_{i \in 1..T} \mathbb{E}_{S \sim D_i^m} L_{D_i}(A(S)) \geq \frac{1}{T} \sum_{i=1}^{T}$$

$$= \frac{1}{T} \sum_{i=1}^{T} \frac{1}{k} \sum_{j=1}^{k} L_{D_i}(A(S_j^i))$$

$$\geq \min_{j \in 1..k} \frac{1}{T} \sum_{i=1}^{T} L_{D_i}(A(S_j^i))$$

$$L_{D_i}(A(S_j^i)) = \frac{1}{2m} \sum_{x \in \mathcal{X}} \mathbb{1}[A(S_j^i)(x) \neq f_i(x)]$$

Fix j, $S_j = (x_1, \dots x_m)$

$\mathcal{X} \setminus S_j = \{ v_1 \dots v_p \}$

$|S_j| \leq m \Rightarrow p \geq m$

$$\geq \frac{1}{2m} \sum_{r=1}^{p} \mathbb{1}[A(S_j^i)(v_r) \neq f_i(v_r)]$$

$$\geq \frac{1}{2p} \sum_{r=1}^{p} \mathbb{1}[A(S_j^i)(v_r) \neq f_i(v_r)]$$

$$\geq \min_{j} \frac{1}{T} \sum_{i=1}^{T} \frac{1}{2p} \sum_{r=1}^{p} \mathbb{1}[A(S_j^i)(v_r) \neq f_i(v_r)]$$

$$\geq \min_{j} \frac{1}{2p} \sum_{r=1}^{p} \frac{1}{T} \sum_{i=1}^{T} \mathbb{1}[A(S_j^i)(v_r) \neq f_i(v_r)]$$

$$\geq \min_{j} \frac{1}{2p} \sum_{r=1}^{p} \frac{1}{T} (T/2)$$

$$\geq \min_{j} \frac{1}{2p} p \times \frac{1}{2} \geq \frac{1}{4}$$